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CORRIGENDUM TO "DEGENERATE SKLYANIN ALGEBRAS AND GENERALIZED TWISTED HOMOGENEOUS COORDINATE RINGS", J. ALGEBRA 322 (2009) 2508-2527

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Abstract. There is an error in the computation of the truncated point schemes V_d of the degenerate Sklyanin algebra $S(1,1,1)$. We are grateful to S. Paul Smith for pointing out that V_d is larger than was claimed in Proposition 3.13. All 2 or 3 digit references are to the above paper, while 1 digit references are to the results in this corrigendum. We provide a description of the correct V_d in Proposition 5 below. Results about the corresponding point parameter ring B associated to the schemes $\{V_d\}_{d\geq 1}$ are given afterward.

1. Corrections

The main error in the above paper is to the statement of Lemma 3.10. Before stating the correct version, we need some notation.

Notation. Given $\zeta = e^{2\pi i/3}$, let $p_a := [1:1:1]$, $p_b := [1:\zeta:\zeta^2]$, and $p_c := [1:\zeta^2:\zeta]$. Also, let $\check{\mathbb{P}}^1_A := \mathbb{P}^1_A \setminus \{p_b, p_c\}$, $\check{\mathbb{P}}^1_B := \mathbb{P}^1_B \setminus \{p_a, p_c\}$, and $\check{\mathbb{P}}^1_C := \mathbb{P}^1_C \setminus \{p_a, p_b\}$.

We also require the following more precise version of Lemma 3.9; the original result is correct though there is a slight change in the proof as given below.

Lemma 1. *(Correction of Lemma 3.9) Let $p = (p_0, \dots, p_{d-2}) \in V_{d-1}$ with $p_{d-2} \in \check{\mathbb{P}}^1_A$, $\check{\mathbb{P}}^1_B$, or $\check{\mathbb{P}}^1_C$. If $p' = (p, p_{d-1}) \in V_d$, then $p_{d-1} = p_a$, p_b, or p_c respectively.*

Proof. The proof follows from that of Lemma 3.9, except that there is a typographical error in the case when $p_{d-2} = [0 : y_{d-2} : z_{d-2}]$. Here, we require that (p_{d-2}, p_{d-1}) satisfies the system of equations:

$$\begin{aligned} f_{d-2} = g_{d-2} = h_{d-2} &= 0, \\ y_{d-2}^3 + z_{d-2}^3 &= 0, \\ x_{d-1}^3 + y_{d-1}^3 + z_{d-1}^3 - 3x_{d-1}y_{d-1}z_{d-1} &= 0. \end{aligned}$$

This implies that either $y_{d-2} = z_{d-2} = 0$ or $x_{d-1} = y_{d-1} = z_{d-1} = 0$, which produces a contradiction. □

Now the correct version of Lemma 3.10 is provided below. The present version is slightly weaker than the original result, where it was claimed that $p_{d-1} \in \check{\mathbb{P}}^1_*$ instead of $p_{d-1} \in \mathbb{P}^1_*$. Here, $\mathbb{P}^1_*$ denotes either $\mathbb{P}^1_A$, $\mathbb{P}^1_B$, or $\mathbb{P}^1_C$.

Lemma 2. *(Correction of Lemma 3.10) Let $p = (p_0, \dots, p_{d-2}) \in V_{d-1}$ with $p_{d-2} = p_a$, p_b, or p_c. If $p' = (p, p_{d-1}) \in V_d$, then $p_{d-1} \in \mathbb{P}^1_A$, $\mathbb{P}^1_B$, or $\mathbb{P}^1_C$ respectively.*

Proof. The proof follows from that of Lemma 3.10 with the exception that there is a typographical error in the definition of the function θ; it should be defined as:

$$\theta(y_{d-1}, z_{d-1}) = \begin{cases} -(y_{d-1} + z_{d-1}) & \text{if } p_{d-2} = p_a, \\ -(\zeta^2 y_{d-1} + \zeta z_{d-1}) & \text{if } p_{d-2} = p_b, \\ -(\zeta y_{d-1} + \zeta^2 z_{d-1}) & \text{if } p_{d-2} = p_c. \end{cases}$$

□

Remark 3. There are two further minor typographical corrections to the paper.

(1) (Correction of Figure 3.1) The definition of the projective lines $\mathbb{P}^1_B$ and $\mathbb{P}^1_C$ should be interchanged. More precisely, the curve E_{111} is the union of three projective lines:

$$\mathbb{P}^1_A : x + y + z = 0,$$
$$\mathbb{P}^1_B : x + \zeta^2 y + \zeta z = 0,$$
$$\mathbb{P}^1_C : x + \zeta y + \zeta^2 z = 0.$$

(2) (Correction to Corollary 4.10) The numbers 57 and 63 should be replaced by 24 and 18 respectively.

2. Consequences

The main consequence of weakening Lemma 3.10 to Lemma 3 is that the truncated point schemes $\{V_d\}_{d\geq 1}$ of $S = S(1,1,1)$ are strictly larger than the truncated point schemes computed in Proposition 3.13 for $d \geq 4$. We discuss such results in §2.1 below. Furthermore, the corresponding point parameter ring associated to the correct point scheme data of S is studied in §2.2.

Notation. (i) Let $W_d := \bigcup_{i=1}^{6} W_{d,i}$ with $W_{d,i}$ defined in Proposition 3.13.
(ii) Let $B := \bigoplus_{d\geq 0} H^0(V_d, \mathcal{O}_{V_d}(\mathbf{1}))$ be the point parameter ring of $S(1,1,1)$ as in Definition 1.8.
(iii) Likewise let $P := \bigoplus_{d\geq 0} H^0(W_d, \mathcal{O}_{W_d}(\mathbf{1}))$ be the point parameter ring associated to the schemes $\{W_d\}_{d\geq 1}$.

The results of §4 of the paper are still correct; we describe the ring P, and we show that it is a factor of $S(1,1,1)$. Unfortunately, the ring P is not equal to the point parameter ring B of $S(1,1,1)$. More precisely, the following corrections should be made.

Remark 4. (1) The scheme V_d should be replaced by W_d in Theorem 1.7, in Proposition 3.13, in Remark 3.14, and in all §4 after Definition 4.1.

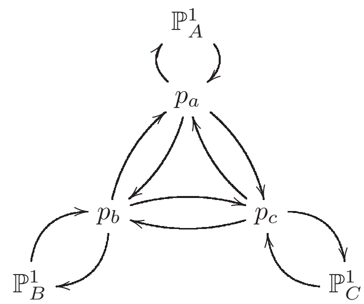
(2) The ring B should be replaced by P in §1 after Definition 1.8, and in all §4 with the exception of the second paragraph.

2.1. On the truncated point schemes $\{V_d\}_{d\geq 1}$.

We provide a description of the truncated point schemes $\{V_d\}_{d\geq 1}$ as follows.

Notation. Let $\{V_{d,i}\}_{i\in I_d}$ denote the $|I_d|$ irreducible components of the d^{th} truncated point scheme V_d.

Proposition 5. *(Description of V_d) For $d \geq 2$, the length d truncated point scheme V_d is realized as the union of length d paths of the quiver Q below. With $d = 2$, for example, the path $\mathbb{P}^1_A \longrightarrow p_a$ corresponds to the component $\mathbb{P}^1_A \times p_a$ of V_2.*



The quiver Q

Proof. We proceed by induction. Considering the $d = 2$ case, Lemma 3.12 still holds so $V_2 = W_2$, the union of the irreducible components:

$$\begin{array}{lll} \mathbb{P}^1_A \times p_a, & \mathbb{P}^1_B \times p_b, & \mathbb{P}^1_C \times p_c \\ p_a \times \mathbb{P}^1_A, & p_b \times \mathbb{P}^1_B, & p_c \times \mathbb{P}^1_C. \end{array}$$

One can see these components correspond to length 2 paths of the quiver Q. Conversely, any length 2 path of Q corresponds to a component that lies in V_2.

We assume the proposition holds for V_{d-1}, and recall that Lemmas 2 and 3 provide the recipe to build V_d from V_{d-1}. Take a point $(p_0, \dots, p_{d-2}) \in V_{d-1,i}$, where the irreducible component $V_{d-1,i}$ of V_{d-1} corresponds to a length $d-1$ path of Q. Let $\{V_{d,ij}\}_{j \in J}$ be the set of $|J|$ irreducible components of V_d with

$$(p_0, \dots, p_{d-2}, p_{d-1}) \in V_{d,ij} \subseteq V_d$$

for some $p_{d-1} \in \mathbb{P}^2$. There are two cases to consider.

<u>Case 1</u>: We have that (p_{d-3}, p_{d-2}) lies in one of the following products:

$$\begin{array}{lll} \mathbb{P}^1_A \times p_a, & \mathbb{P}^1_B \times p_b, & \mathbb{P}^1_C \times p_c, \\ p_a \times \check{\mathbb{P}}^1_A, & p_b \times \check{\mathbb{P}}^1_B, & p_c \times \check{\mathbb{P}}^1_C. \end{array}$$

For the first three choices, Lemma 2 implies that $pr_d(V_{d,ij}) = \mathbb{P}^1_A$, $\mathbb{P}^1_B$, or $\mathbb{P}^1_C$, respectively. For the second three choices, p_{d-2} belongs to $\check{\mathbb{P}}^1_A$, $\check{\mathbb{P}}^1_B$, or $\check{\mathbb{P}}^1_C$, and Lemma 1 implies that $pr_d(V_{d,ij}) = p_a$, p_b, or p_c, respectively. We conclude by induction that the component $V_{d,ij}$ yields a length d path of Q.

<u>Case 2</u>: We have that (p_{d-3}, p_{d-2}) is equal to one of the following points:

$$\begin{array}{ll} p_a \times p_b, & p_a \times p_c, \\ p_b \times p_a, & p_b \times p_c, \\ p_c \times p_a, & p_c \times p_b. \end{array}$$

Now Lemma 2 implies that:

$$pr_d(V_{d,ij}) = \begin{cases} \mathbb{P}^1_A & \text{if } p_{d-2} = p_a, \\ \mathbb{P}^1_B & \text{if } p_{d-2} = p_b, \\ \mathbb{P}^1_C & \text{if } p_{d-2} = p_c. \end{cases}$$

Again we have that in this case, the component $V_{d,ij}$ yields a length d path of Q.

Conversely (in either case), let $\mathcal{P}$ be a length d path of Q. Then, by induction, the embedded length $d-1$ path $\mathcal{P}'$ ending at the $d-1^{\text{st}}$ vertex v' of $\mathcal{P}$ yields a component X' of V_{d-1}. Say v is the d^{th} vertex of $\mathcal{P}$. If v' is equal to $\mathbb{P}^1_A$, $\mathbb{P}^1_B$, or $\mathbb{P}^1_C$, then v must be p_a, p_b, or p_c by the definition of Q, respectively. Lemma 2 then ensures that $\mathcal{P}$ yields a component X of V_d so that $pr_{1\ldots d-1}(X) = X'$. On the other hand, if v' is equal to p_a, p_b, or p_c, then v lies in $\mathbb{P}^1_A$, $\mathbb{P}^1_B$, or $\mathbb{P}^1_C$, respectively. Likewise, Lemma 3 implies that $\mathcal{P}$ yields a component X of V_d so that $pr_{1\ldots d-1}(X) = X'$. □

Corollary 6. *We have that $V_d = W_d$ for $d = 1, 2, 3$, and that $V_d \supsetneq W_d$ for $d \geq 4$.*

Proof. First, $V_1 = \mathbb{P}^2 = W_1$. Next, as mentioned in the proof of Proposition 5, $V_2 = W_2$ is the union of the irreducible components:

$$\begin{array}{lll} \mathbb{P}^1_A \times p_a, & \mathbb{P}^1_B \times p_b, & \mathbb{P}^1_C \times p_c \\ p_a \times \mathbb{P}^1_A, & p_b \times \mathbb{P}^1_B, & p_c \times \mathbb{P}^1_C. \end{array}$$

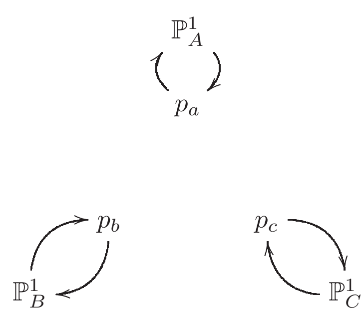
By Proposition 5, we have that $V_3 = X_{3,1} \cup X_{3,2}$ where $X_{3,1}$ consists of the irreducible components:

$$\begin{array}{lll} \mathbb{P}^1_A \times p_a \times \mathbb{P}^1_A, & \mathbb{P}^1_B \times p_b \times \mathbb{P}^1_B, & \mathbb{P}^1_C \times p_c \times \mathbb{P}^1_C, \\ p_a \times \mathbb{P}^1_A \times p_a, & p_b \times \mathbb{P}^1_B \times p_b, & p_c \times \mathbb{P}^1_C \times p_c, \end{array}$$

and $X_{3,2}$ is the union of:

$$\begin{array}{llll} \mathbb{P}^1_A \times p_a \times p_b, & \mathbb{P}^1_A \times p_a \times p_c, & p_a \times p_b \times \mathbb{P}^1_B, & p_a \times p_c \times \mathbb{P}^1_C, \\ p_a \times p_b \times p_a, & p_a \times p_b \times p_c, & p_a \times p_c \times p_a, & p_a \times p_c \times p_b, \\ \mathbb{P}^1_B \times p_b \times p_c, & \mathbb{P}^1_B \times p_b \times p_a, & p_b \times p_c \times \mathbb{P}^1_C, & p_b \times p_a \times \mathbb{P}^1_A, \\ p_b \times p_c \times p_b, & p_b \times p_c \times p_a, & p_b \times p_a \times p_b, & p_b \times p_a \times p_c, \\ \mathbb{P}^1_C \times p_c \times p_a, & \mathbb{P}^1_C \times p_c \times p_b, & p_c \times p_a \times \mathbb{P}^1_A, & p_c \times p_b \times \mathbb{P}^1_B, \\ p_c \times p_a \times p_c, & p_c \times p_a \times p_b, & p_c \times p_b \times p_c, & p_c \times p_b \times p_a. \end{array}$$

Note that $X_{3,2}$ is contained in $X_{3,1}$; hence $V_3 = X_{3,1} = W_3$. Furthermore, one sees that $W_d \subsetneq V_d$ for $d \geq 4$ as follows. The components of W_d are read off the subquiver Q' of Q below.



The quiver Q'

On the other hand, for $d \geq 4$, the length d path containing

$$\mathbb{P}^1_A \longrightarrow p_a \longrightarrow p_b \longrightarrow \mathbb{P}^1_B$$

corresponds to a component of V_d not contained in W_d. □

2.2. **On the point parameter ring** $B(\{V_d\})$**.** The result that there exists a ring surjection from $S = S(1,1,1)$ onto the ring $P(\{W_d\})$ remains true. However, by Lemma 7 below, B is a larger ring than P, and whether there is a ring surjection from S onto B is unknown. We know that there is a ring homomorphism from S to B with $S_1 \cong B_1$ by [1, Proposition 3.20], and computational evidence suggests that $S \cong B$. The details are given as follows.

Lemma 7. *The k-vector space dimension of B_d is equal to* $\dim_k S(1,1,1)_d$ *for $d = 0, 1, \ldots, 4$. In particular,* $\dim_k B_4 \neq \dim_k P_4$.

It is believed that analogous computations will show that $\dim_k B_d = \dim_k S(1,1,1)_d = 3 \cdot 2^{d-1}$ for $d = 5, 6$.

Proof of Lemma 7. By Corollary 6, we know that $V_d = W_d$ for $d = 1, 2, 3$; hence

$$\dim_k B_d = 3 \cdot 2^{d-1} = \dim_k S(1,1,1)_d \text{ for } d = 0, 1, 2, 3.$$

To compute $\dim_k B_4$, note that by Proposition 5, V_4 equals the union $X_{4,1} \cup X_{4,2} \subseteq (\mathbb{P}^2)^{\times 4}$ as follows. Here, $X_{4,1}$ consists of the following irreducible components

$$\begin{array}{ll}
\mathbb{P}^1_A \times p_a \times \mathbb{P}^1_A \times p_a, & p_a \times \mathbb{P}^1_A \times p_a \times \mathbb{P}^1_A, \\
\mathbb{P}^1_B \times p_b \times \mathbb{P}^1_B \times p_b, & p_b \times \mathbb{P}^1_B \times p_b \times \mathbb{P}^1_B, \\
\mathbb{P}^1_C \times p_c \times \mathbb{P}^1_C \times p_c, & p_c \times \mathbb{P}^1_C \times p_c \times \mathbb{P}^1_C;
\end{array}$$

and $X_{4,2}$ is the union of

$$\begin{array}{ll}
\mathbb{P}^1_A \times p_a \times p_b \times \mathbb{P}^1_B, & \mathbb{P}^1_A \times p_a \times p_c \times \mathbb{P}^1_C, \\
\mathbb{P}^1_B \times p_b \times p_a \times \mathbb{P}^1_A, & \mathbb{P}^1_B \times p_b \times p_c \times \mathbb{P}^1_C, \\
\mathbb{P}^1_C \times p_c \times p_a \times \mathbb{P}^1_A, & \mathbb{P}^1_C \times p_c \times p_b \times \mathbb{P}^1_B.
\end{array}$$

We consider a component such as $\mathbb{P}^1_A \times p_a \times p_b \times p_a$ contained in $\mathbb{P}^1_A \times p_a \times p_b \times \mathbb{P}^1_B$ to be included as part of $X_{4,2}$.

Since $X_{4,1} = W_4$ we get that $h^0(\mathcal{O}_{X_{4,1}}(1,1,1,1)) = 6 \cdot 4 - 6 = 18$ by Proposition 4.3. Moreover, $h^0(\mathcal{O}_{X_{4,2}}(1,1,1,1)) = 6 \cdot 4 = 24$ as $X_{4,2}$ is a disjoint union of its irreducible components.

Consider the finite morphism

$$\pi_1 : X_{4,1} \uplus X_{4,2} \longrightarrow V_4 = X_{4,1} \cup X_{4,2},$$

which by twisting by $\mathcal{O}_{(\mathbb{P}^2)^{\times 4}}(1,1,1,1)$, we get the exact sequence:

$$\begin{aligned}
0 \longrightarrow \mathcal{O}_{V_4}(1,1,1,1) &\longrightarrow [(\pi_1)_* \mathcal{O}_{X_{4,1} \uplus X_{4,2}}](1,1,1,1) \\
&\longrightarrow \mathcal{O}_{X_{4,1} \cap X_{4,2}}(1,1,1,1) \\
&\longrightarrow 0.
\end{aligned} \tag{†}$$

Here, $X_{4,1} \cap X_{4,2}$ is the union of the following irreducible components:

$$\begin{array}{ll}
\mathbb{P}^1_A \times p_a \times p_b \times p_a, & p_b \times p_a \times p_b \times \mathbb{P}^1_B, \\
\mathbb{P}^1_A \times p_a \times p_c \times p_a, & p_c \times p_a \times p_c \times \mathbb{P}^1_C, \\
\mathbb{P}^1_B \times p_b \times p_a \times p_b, & p_a \times p_b \times p_a \times \mathbb{P}^1_A, \\
\mathbb{P}^1_B \times p_b \times p_c \times p_b, & p_c \times p_b \times p_c \times \mathbb{P}^1_C, \\
\mathbb{P}^1_C \times p_c \times p_a \times p_c, & p_a \times p_c \times p_a \times \mathbb{P}^1_A, \\
\mathbb{P}^1_C \times p_c \times p_b \times p_c, & p_b \times p_c \times p_b \times \mathbb{P}^1_B,
\end{array}$$

a union that is not disjoint. Let $(X_{4,1} \cap X_{4,2})'$ be the disjoint union of these twelve components and consider the finite morphism

$$\pi_2 : (X_{4,1} \cap X_{4,2})' \to X_{4,1} \cap X_{4,2}.$$

Again by twisting by $\mathcal{O}_{\mathbb{P}^2}(1,1,1,1)$, we get the exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1) &\longrightarrow [(\pi_2)_*\mathcal{O}_{(X_{4,1}\cap X_{4,2})'}](1,1,1,1) \\ &\longrightarrow \mathcal{O}_{\mathcal{S}}(1,1,1,1) \\ &\longrightarrow 0, \end{aligned} \tag{‡}$$

where $\mathcal{S}$ is the union of the following six points:

$$\begin{array}{lll} p_a \times p_b \times p_a \times p_b, & p_b \times p_a \times p_b \times p_a, & p_a \times p_c \times p_a \times p_c, \\ p_c \times p_a \times p_c \times p_a, & p_b \times p_c \times p_b \times p_c, & p_c \times p_b \times p_c \times p_b. \end{array}$$

Claim 1. $H^1(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1)) = 0.$

Note that $H^0([(\pi_2)_*\mathcal{O}_{(X_{4,1}\cap X_{4,2})'}](1,1,1,1)) \cong H^0(\mathcal{O}_{(X_{4,1}\cap X_{4,2})'}(1,1,1,1))$ as k-vector spaces since π_2 is an affine map [2, Exercise III 4.1]. Hence, if Claim 1 holds, then by (‡):

$$\begin{aligned} h^0(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1)) &= h^0(\mathcal{O}_{(X_{4,1}\cap X_{4,2})'}(1,1,1,1)) - h^0(\mathcal{O}_{\mathcal{S}}(1,1,1,1)) \\ &= 12 \cdot 2 - 6 = 18. \end{aligned}$$

Claim 2. $H^1(\mathcal{O}_{V_4}(1,1,1,1)) = 0.$

Note that $H^0([(\pi_1)_*\mathcal{O}_{X_{4,1}\uplus X_{4,2}}](1,1,1,1)) \cong H^0(\mathcal{O}_{X_{4,1}\uplus X_{4,2}}(1,1,1,1))$ as k-vector spaces since π_1 is an affine map [2, Exercise III 4.1]. Hence, if Claim 2 is also true, then by (†) and the computation above, we note that:

$$\begin{aligned} \dim_k B_4 &= h^0(\mathcal{O}_{V_4}(1,1,1,1)) \\ &= h^0(\mathcal{O}_{X_{4,1}\uplus X_{4,2}}(1,1,1,1)) - h^0(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1)) \\ &= h^0(\mathcal{O}_{X_{4,1}}(1,1,1,1)) + h^0(\mathcal{O}_{X_{4,2}}(1,1,1,1)) - h^0(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1)) \\ &= 18 + 24 - 18 = 24. \end{aligned}$$

Therefore,

$$\dim_k B_4 = \dim_k S(1,1,1)_4 = 24 \neq 18 = \dim_k P_4.$$

Now we prove Claims 1 and 2 above. Here, we refer to the linear components of $(\mathbb{P}^2)^{\times 4}$ of dimensions 1 or 2 by "lines" or "planes", respectively.

Proof of Claim 1. It suffices to show that

$$\theta : H^0\left(\mathcal{O}_{(X_{4,1}\cap X_{4,2})'}(1,1,1,1)\right) \longrightarrow H^0(\mathcal{O}_{\mathcal{S}}(1,1,1,1))$$

is surjective. Say $\mathcal{S} = \{v_i\}_{i=1}^6$, the union of points v_i. Each point v_i is contained in two lines of $(X_1 \cap X_2)'$, and each of the twelve lines of $(X_1 \cap X_2)'$ contains a unique point of $\mathcal{S}$.

Choose a basis $\{t_i\}_{i=1}^6$ for $H^0(\mathcal{S}(1,1,1,1))$, where $t_i(v_j) = \delta_{ij}$. For each i, there exists a unique line L_i of $(X_{4,1} \cap X_{4,2})'$ containing v_i so that $pr_{234}(L_i) = pr_{234}(v_i)$. Now we define a preimage of t_i by first extending t_i to a global section s_i of $\mathcal{O}_{L_i}(1,1,1,1)$. Moreover, extend s_i to a global section

$\tilde{s_i}$ on $\mathcal{O}_{(X_{4,1}\cap X_{4,2})'}(1,1,1,1)$ by declaring that $\tilde{s_i} = s_i$ on L_i and zero elsewhere. Now $\theta(\tilde{s_i}) = t_i$ for all i, and θ is surjective. □

Proof of Claim 2. It suffices to show that

$$\tau : H^0(\mathcal{O}_{X_{4,1}\uplus X_{4,2}}(1,1,1,1)) \longrightarrow H^0(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1))$$

is surjective.

Recall that $X_{4,1} \cap X_{4,2}$ is the union of twelve lines $\{L_i\}$, and $X_{4,1} \uplus X_{4,2}$ is the union of twelve planes $\{P_i\}$. Here, each line L_i of $X_{4,1} \cap X_{4,2}$ is contained in precisely two planes of $X_{4,1} \uplus X_{4,2}$, and each plane P_i of $X_{4,1} \uplus X_{4,2}$ contains precisely two lines of $X_{4,1} \cap X_{4,2}$.

Choose a basis $\{t_i\}_{i=1}^{12}$ of $H^0\left(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1)\right)$ so that $t_i(L_j) = \delta_{ij}$. For each i, we want a preimage of t_i in $H^0\left(\mathcal{O}_{X_{4,1}\uplus X_{4,2}}(1,1,1,1)\right)$.

Say P_i is a plane of $X_{4,1} \uplus X_{4,2}$ that contains L_i, and L_j is the other line that is contained in P_i. Since $\mathcal{O}_{P_i}(1,1,1,1)$ is very ample, its global sections separate the lines L_i and L_j. In other words, there exists $s_i \in H^0(\mathcal{O}_{P_i}(1,1,1,1))$ so that $s_i(L_k) = \delta_{ik}$. Extend s_i to $\tilde{s_i} \in H^0\left(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1)\right)$ by declaring that $\tilde{s_i} = s_i$ on L_i, and zero elsewhere. Now $\tau(\tilde{s_i}) = t_i$ for all i, and τ is surjective. □

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